BILINEAR STRICHARTZ ESTIMATES FOR SCHRÖDINGER OPERATORS IN 2 DIMENSIONAL COMPACT MANIFOLDS WITH BOUNDARY AND CUBIC NLS

JIN-CHENG JIANG

Abstract. In this paper, we establish bilinear and gradient bilinear Strichartz estimates for Schrödinger operators in 2 dimensional compact manifolds with boundary. Using these estimates, we can infer the local well-posedness of cubic nonlinear Schrödinger equation in H^s for every $s > \frac{2}{3}$ on such manifolds.

1. Introduction and Results

Let (M, g) be a Riemannian manifold of dimension $n \geq 2$. Consider the Schrödinger equation

$$D_t u + \Delta_g u = 0, \qquad u(0, x) = f(x) \tag{1.1}$$

where Δ_g denotes the Laplace-Beltrami operator on manifold and $D_t = i^{-1}\partial_t$. Strichartz estimates are a family of dispersive estimates on solutions $u(t,x) : [0,T] \times M \to \mathbb{C}$ which state

$$\|u\|_{L^p([0,T];L^q(M))} \leq C\|f\|_{H^s(M)} \tag{1.2}$$

where H^s denotes the L^2 Sobolev space over M, and $2 \leq p, q \leq \infty$ satisfies

$$\frac{2}{p} + \frac{n}{q} = \frac{n}{2} \qquad (n, p, q) \neq (2, 2, \infty).$$

In Euclidean space, one can take $T = \infty$ and $s = 0$; see for example Strichartz [22], Ginibre and Velo [14], Keel and Tao [16] and references therein. Such estimates have been a key tool in the study of nonlinear Schrödinger equations. In the case of compact manifolds (M, g) without boundary Burq, Gérard and Tzvetkov [11] proved the finite time scale estimates (1.2) for the Schrödinger operators with a loss of derivatives $s = \frac{1}{p}$ in their estimates when compared to the case of flat geometries.

In the case of compact manifolds with boundary, one considers Dirichlet or Neumann boundary conditions in addition to (1.1)

$$u(t,x)|_{\partial M} = 0 \text{ (Dirichlet)}, \quad \text{or} \quad N_x \cdot \nabla u(t,x)|_{\partial M} = 0 \text{ (Neumann)}$$

where N_x denotes the unit normal vector field to ∂M. Here one excepts a further loss of derivatives due to Rayleigh whispering galley modes. Recently, Anton [4]

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showed that the estimates (1.2) hold on general manifolds with boundary if $s > \frac{3}{2p}$ which arguments of [4] work equally well for a manifold without boundary equipped with a Lipschitz metric. Then Blair, Smith and Sogge [5] built estimates (1.2) with a less loss of derivatives $s = \frac{4}{3p}$ in manifolds with boundary.

Write $u = e^{it\Delta} f$ as the solution of (1.1) with initial data f. We consider bilinear estimates for the Schrödinger operators in compact manifolds of the form

$$\|e^{it\Delta} f e^{it\Delta} g\|_{L^2([0,1]\times M)} \le C(\min(\Lambda, \Gamma))^{s_0} \|f\|_{L^2(M)} \|g\|_{L^2(M)} \tag{1.3}$$

where Λ, Γ are large dyadic numbers, and f, g are supposed to be spectrally localized on dyadic intervals of order Λ, Γ respectively, namely

$$\mathbb{I}_{\Lambda \le \sqrt{-\Delta} \le 2\Lambda}(f) = f \quad , \quad \mathbb{I}_{\Gamma \le \sqrt{-\Delta} \le 2\Gamma}(g) = g.$$

Here $\mathbb{I}_{\Lambda \le \sqrt{-\Delta} \le 2\Lambda}$ denotes the spectral projection operator

$$\sum_{\Lambda \le \Lambda_j \le 2\Lambda} E_j f = \sum_{\Lambda \le \Lambda_j \le 2\Lambda} e_j \int_M f e_j \ ,$$

while $\{\Lambda_j^2\}$ and $\{e_j\}$ are eigenvalues and corresponding eigenfunctions of $-\Delta_g$. Such kind of estimates were established and used on Schrödinger equation on manifolds with flat metric; see Klainerman-Machedon-Bourgain-Tataru [17], Bourgain [6] and Tao [23] and reference therein . Then Burq, Gérard and Tzvetkov [12] established the bilinear estimates in sphere and Zoll surfaces with $s_0 > \frac{1}{4}$. In the cases of sphere and Zoll surfaces [12] , due to the good locations of eigenvalues for the Laplacian, the bilinear Strichartz estimates are reduced to bilinear spectral cluster estimates. For general manifolds, our poor knowledge of spectrums does not allow us to use the same technique. One of our main results here is showing that by considering the endpoint of admissible pairs for the Schrödinger operator and using the parametrix construction, we can get the bilinear Strichartz estimates for general 2 dimensional manifolds, though the estimates are not known to be sharp.

Consider Strichartz estimates on manifolds with boundary obtained by Blair, Smith and Sogge [5]. When $n = 2$, $(p, q) = (4, 4)$ is admissible, so we have

$$\|e^{it\Delta} f\|_{L^4([0,1]\times M)} \le C\|f\|_{H^{1/3}(M)}.$$

Using Littlewood-Paley theory, let $f_\Lambda = \mathbb{I}_{\Lambda \le \sqrt{-\Delta} \le 2\Lambda}(f)$, this is equivalent to say $\|e^{it\Delta} f_\Lambda\|_{L^4([0,1]\times M)} \le C\Lambda^{1/3} \|f_\Lambda\|_{L^2(M)}$ holds for all dyadic number Λ, which is implied by bilinear estimates (1.3) with $s_0 = \frac{2}{3}$. However we would establish the following estimates with $s_0 > \frac{2}{3}$.

Theorem 1.1. *Let (M, g) be a 2 dimensional compact manifold with boundary. For any $f,\ g \in L^2(M)$ satisfies*

$$\mathbb{I}_{\Lambda \le \sqrt{-\Delta} \le 2\Lambda}(f) = f \quad \mathbb{I}_{\Gamma \le \sqrt{-\Delta} \le 2\Gamma}(g) = g.$$

Then for any $s_0 > \frac{2}{3}$, there exists a $C > 0$ such that

$$\|e^{it\Delta} f e^{it\Delta} g\|_{L^2([0,1]\times M)} \le C(\min(\Lambda, \Gamma))^{s_0} \|f\|_{L^2(M)} \|g\|_{L^2(M)}. \tag{1.4}$$

Remark 1.2. Our proof of Theorem 1.1 can be simplified to get the bilinear Strichartz estimates with $s_0 > \frac{1}{2}$ in 2 dimensional compact manifolds without boundary.

For compact manifold with boundary, Anton [3] proved (1.3) and the following

$$\|(\nabla e^{it\Delta} f)e^{it\Delta} g\|_{L^2([0,1]\times M)} \leq C\Lambda(\min(\Lambda,\Gamma))^{s_0}\|f\|_{L^2(M)}\|g\|_{L^2(M)} \tag{1.5}$$

with $s_0 > \frac{1}{2}$ on three dimensional balls with Dirichlet boundary condition and radial data. She used the same idea as [12], thanks again the good locations of eigenvalues for the Laplacian in such setting. Using (1.3) and (1.5) with $s_0 > \frac{1}{2}$, she proved the local well-posedness of cubic nonlinear Schrödinger equation with Dirichlet boundary condition and radial data in H^s for every $s > \frac{1}{2}$ on three dimensional balls. In order to build the corresponding estimates in our case, we need more results from harmonic analysis besides the parametrix construction of solutions for the free equation. There are two different cases. If the gradient operator is acting on the solution has initial data being localized to the larger frequency, then we can exploit the boundedness of Riesz transform (see [18]) on $L^2(M)$, then apply the Hörmander multiple theorem (for manifold with boundary, see [27]) to get the desired result. For the other case, we make use of Xu's [27] estimates for the gradient spectral cluster operators. Following by an argument concerning the finite propagation speed of solutions to the wave equation (see for example [21], [27]), then we can control the L^2 norm from the estimates of gradient spectral cluster operators by a L^∞ norm, thus return to the parametrix construction argument again.

Our gradient bilinear Strichartz estimate is the following.

Theorem 1.3. *Let (M,g) be a 2 dimensional compact manifold with boundary. For any $f,\ g \in L^2(M)$ satisfies*

$$\mathbb{I}_{\Lambda\leq\sqrt{-\Delta}\leq 2\Lambda}(f) = f \quad \mathbb{I}_{\Gamma\leq\sqrt{-\Delta}\leq 2\Gamma}(g) = g.$$

Then for any $s_0 > \frac{2}{3}$, there exists a $C > 0$ such that

$$\|(\nabla_x(e^{it\Delta} f))e^{it\Delta} g\|_{L^2([0,1]\times M)} \leq C\Lambda(\min(\Lambda,\Gamma))^{s_0}\|f\|_{L^2(M)}\|g\|_{L^2(M)} \tag{1.6}$$

After we establish (1.4) and (1.6) to solutions of (1.1) satisfying either Dirichlet or Neumann boundary conditions for the general 2 dimensional compact manifolds with boundary, we will follow Anton's [3] argument to prove local well-posedness property in our setting.

We consider the following Cauchy problem in 2-dimensional compact manifolds with boundary:

$$\left\{\begin{array}{rcl} i\partial_t u + \triangle u &=& \alpha|u|^2 u, \text{ in } \mathbb{R}\times M \\ u|_{t=0} &=& u_0, \text{ on } M \\ u|_{\partial M} &=& 0 \text{ (Dirichlet)}, \quad \text{(or)} \quad \mathrm{N_x}\cdot\nabla \mathrm{u}|_{\partial\mathrm{M}} = 0 \text{ (Neumann)} \end{array}\right. \tag{1.7}$$

where $\alpha = \pm 1$. When $\alpha = 1$, the equation is defocusing. When $\alpha = -1$, the equation is focusing. We consider the local well-posedness property of (1.7).

Definition 1.4. Let s be a real number. We shall say that the Cauchy problem (1.7) is uniformly well-posed in $H^s(M)$ if, for any bounded subset B of $H^s(M)$, there exists $T > 0$ such that the flow map

$$u_0 \in C^\infty(M)\cap B \mapsto u \in C([-T,T], H^s(M))$$

is uniformly continuous when the source space is endowed with H^s norm, and when the target space is endowed with

$$\|u\|_{C_T H^s} = sup_{|t|\leq T}\|u(t)\|_{H^s(M)}$$

Our discussions in the following focus again in 2 dimensional case. For manifolds without boundary, we only consider first two equations of (1.7). The first result was due to Bourgain [9] who built the local well-posedness result in H^s for $s > 0$ on the flat torus. Recently, Burq, Gérard and Tzvetkov [11] use Strichartz estimates to establish local well-posedness of cubic nonlinear Schrödinger equation in $H^s(M)$ for $s > \frac{1}{2}$ on 2 dimensional manifold without boundary. In [12] they proved the local well-posed property in $H^s(M)$ for $s > \frac{1}{4}$ on sphere and Zoll surface by using the bilinear Strichartz estimates (1.3) with $s_0 > \frac{1}{4}$.

For manifolds with boundary, it is natural to except a more loss of derivatives due to Rayleigh whispering galley modes. In the case of domains of $\mathbb{R}^2$ the local well-posedness for (1.7) with Dirichlet boundary condition and $s = 1$ were proved by Anton [4]. On the other direction, Burq, Gérard and Tzvetkov [10] built an illposedness result on a disc of $\mathbb{R}^2$, for $s < \frac{1}{3}$.

Our result is the following.

Theorem 1.5. *If (M, g) is a 2 dimensional manifold with boundary, then the Cauchy problem (1.7) is uniformly well-posed in $H^s(M)$ for every $s > \frac{2}{3}$.*

2. Reductions

We start with the proof of Theorem 1.1. The Laplace-Beltrami operators on M will take the following form in local coordinates

$$(Pf)(x) = \rho^{-1}\sum_{i,j=1}^{n}\partial_i(\rho(x)g^{ij}(x)\partial_j f(x)) \tag{2.1}$$

Assume $\mathbb{I}_{\Lambda\leq\sqrt{-\Delta}\leq 2\Lambda}(f) = f$, $\mathbb{I}_{\Gamma\leq\sqrt{-\Delta}\leq 2\Gamma}(g) = g$ and $\Lambda < \Gamma$. Then

$$\begin{aligned}\|e^{it\Delta}f\ e^{it\Delta}g\|_{L^2([0,1]\times M)} &\lesssim \|v\|_{L^\infty([0,1];L^2(M))}\|u\|_{L^2([0,1];L^\infty(M))}\\ &\lesssim \|g\|_{L^2(M))}\|u\|_{L^2([0,1];L^\infty(M))},\end{aligned} \tag{2.2}$$

where we have used the conservation of mass for the free Schrödinger operator in the last inequality.

We define Sobolev spaces on M using the spectral resolution of P,

$$\|f\|_{H^s(M)} = \|\langle D_p\rangle^s f\|_{L^2(M)} \ \ , \ \ \langle D_p\rangle = (1-P)^{\frac{1}{2}}$$

By elliptic regularity (e.g [[13], Theorem 8.10]) the space H^s coincide with the Sobolev spaces defined using local coordinates, provided $0 \leq s \leq 2$.

Let $r = \frac{2}{3} + \varepsilon > \frac{2}{3}$, $s = r - 1$. Then we need to establish

$$\|u\|_{L^2([0,1];L^\infty(M))} \lesssim \|f\|_{H^r(M)} \approx (\Lambda)^r\|f\|_{L^2(M)},$$

or equivalently,

$$\|u\|_{L^2([0,1];L^\infty(M))} \lesssim \|\Lambda^s f\|_{H^1(M)}$$

By conservation law of free Schrödinger operator which is equivalent to

$$\|u\|_{L^2([0,1];L^\infty(M))} \lesssim \|\Lambda^s u\|_{L^\infty([0,1];H^1(M))} \tag{2.3}$$

Although $(2,2,\infty)$ is not Schrödinger admissible, we should see that once we localize both time and frequency we can still get desired type of Strichartz estimates.

We work in boundary normal coordinates for the Riemannian metric g_{ij} that is dual g^{ij} of (2.1). Let $x_2 > 0$ define the manifold M, and x_1 is a coordinate function on ∂M which we choose so that ∂_{x_1} is of unit length along ∂M. In these coordinates,

$$g_{22}(x_1,x_2) = 1 \quad g_{11}(x_1,0) = 1 \quad g_{12}(x_1,x_2) = 0$$

We now extend the coefficient g^{11} and ρ in an even manner across the boundary, so that

$$g^{11}(x_1,-x_2) = g^{11}(x_1,x_2) \quad \rho(x_1,-x_2) = \rho(x_1,x_2).$$

The extended functions are then piecewise smooth, and of Lipschitz regularity across $x_2 = 0$. Because g is diagonal, the operator P is preserved under the reflection $x_2 \to -x_2$. Eigenspaces for the extended operator $\tilde{P}$ decompose into symmetric and antisymmetric functions; these correspond to extensions of eigenfunctions for P satisfying Dirichlet (resp. Neumann) conditions. These eigenfunctions are of $C^{1,1}$ across the boundary. The Schrödinger flow for P is thus extended to $\tilde{P}$.

Hence matters reduces to considering the Schrödinger evolution on the manifold without boundary with Lipschitz metrics. And we have to show

$$\|u\|_{L^2([0,1],L^\infty(M))} \lesssim \|\Lambda^s u\|_{L^\infty([0,1];H^1(M))}$$

By taking a finite partition of unity, it suffices to prove that

$$\|\psi u\|_{L^2([0,1];L^\infty(\mathbb{R}^2))} \lesssim \|\Lambda^s u\|_{L^\infty([0,1];H^1(M))}$$

for each smooth cutoff ψ supported in a suitably chosen coordinate charts. We will choose coordinate charts such that the image contains the unit ball, and

$$\|g^{ij} - \delta_{ij}\|_{Lip(B_1(0))} \le c_0 \ , \ \|\rho - 1\|_{Lip(B_1(0))} \le c_0$$

for c_0 to be taken suitably small. We take ψ supported in the unit ball, and assume g^{ij} and ρ are extended so that the above holds globally on $\mathbb{R}^2$.

We denote $u = u_k$ to address that it's frequency being localized to $\Lambda = 2^k$, the estimates we need is now

$$\|\psi u_k\|_{L^2([0,1];L^\infty(\mathbb{R}^2))} \lesssim \|\Lambda^s u_k\|_{L^\infty([0,1];H^1(M))}.$$

Let $\{\beta_j(D)\}_{j\ge 0}$ be a Littlewood-Paley partition of unity on $\mathbb{R}^n$, and let $v_j = \beta_j(D)(\psi u_k)$, $v_j^s = (2^j)^s v_j$, then we will see that it is equivalent to show that for each j,

$$\|v_j\|_{L^2_t L^\infty_x} \lesssim \|v_j^s\|_{L^\infty_t H^1_x} + (2^j)^{s-1/3}\|(D_t + P)v_j\|_{L^\infty_t L^2_x}) \tag{2.4}$$

is true, where the norm is taken over $(t,x) = [0,1] \times \mathbb{R}^2$. Note that for any $\varepsilon > 0$

$$\|\psi u_k\|_{L^2_t L^\infty_x} \lesssim \|2^{j\varepsilon} v_j\|_{L^2_t L^\infty_x l^j_2} \lesssim \|2^{j\varepsilon} v_j\|_{l^j_2 L^2_t L^\infty_x}.$$

Here ε can be absorbed by s in (2.4), thus we only have to deal with $\|v_j\|$ instead of $\|2^{j\varepsilon}v_j\|$ in (2.4).

On the other hand,

$$\begin{aligned}\|v_j^s\|_{L^\infty([0,1];H^1(\mathbb{R}^2))} &\lesssim \min\{(2^j)\|v_j^s\|_{L^\infty([0,1];L^2(\mathbb{R}^2))}, (2^j)^{-1}\|v_j^s\|_{L^\infty([0,1];H^2(\mathbb{R}^2))}\} \\ &\lesssim \min\{(2^j)^{1+s}\|u_k\|_{L^\infty([0,1];L^2(M))}, (2^j)^{-1+s}\|u_k\|_{L^\infty([0,1];H^2(M))}\}\end{aligned}$$

To sum up $\|v_j^s\|_{L_t^\infty H_x^1}$ over j, we dominate those terms with $j \le k$ by the first term inside minimum bracket, dominate those terms with $j \ge k$ by the second term inside minimum bracket. The series is then bounded by a finite sum plus a geometric series. So the summation over j of first terms in the right hand side of (2.4) is bounded by

$$\begin{aligned}(2^k)^{1+s}\|u_k\|_{L^\infty([0,1];L^2(M))} + (2^k)^{-1+s}\|u_k\|_{L^\infty([0,1];H^2(M))} &\lesssim (2^k)^s\|u_k\|_{L^\infty([0,1];H^1(M))} \\ &\approx \|\Lambda^s u_k\|_{L^\infty([0,1];H^1(M))}\end{aligned}$$

For the second term in the right hand side of (2.4), we note that for a Lipschitz function a, $[\beta_j(D), a] : H^{s-1} \to H^s$, $s = 0, 1$. Hence $[P, \beta_j(D)\psi] : H^1 \to L^2$, by Coifman-Meyer commutator theorem (see also Proposition 3.6B of [26]). Therefore we have

$$\|(D_t + P)v_j\|_{L^\infty([0,1];L^2(\mathbb{R}^2))} \lesssim \|u_k\|_{L^\infty([0,1]\ H^1(M))}. \tag{2.5}$$

Furthermore, we claim that the following estimate is also true

$$\|(D_t + P)v_j\|_{L^\infty([0,1];L^2(\mathbb{R}^2))} \lesssim 2^j\|u_k\|_{L^\infty([0,1]\ L^2(M))} \tag{2.6}$$

First, we truncate the coefficients of P to frequencies less than some small constant times $2^j = \eta$ and denote the new coefficients and operator by g_η^{ij} and P_j respectively. Note that the localized coefficients satisfy $|g^{ij} - g_\eta^{ij}| \lesssim 2^{-j}$. Thus

$$\|(P_j - P)v_j\|_{L^\infty([0,1];L^2(\mathbb{R}^2))} \lesssim 2^j\ \|v_j\|_{L^\infty([0,1];L^2(\mathbb{R}^2))} \lesssim 2^j\ \|u_k\|_{L^\infty([0,1];L^2(M))}\,. \tag{2.7}$$

Combine this with

$$\begin{aligned}&\|(D_t + P)v_j\|_{L^\infty[0,1];L^2(\mathbb{R}^2)} \\ &\le \|(D_t + P_j)v_j\|_{L^\infty[0,1];L^2(\mathbb{R}^2)} + \|(P - P_j)v_j\|_{L^\infty[0,1];L^2(\mathbb{R}^2)}\,,\end{aligned}$$

we are reduced to estimate

$$\|(D_t + P_j)v_j\|_{L^\infty[0,1];L^2(\mathbb{R}^2)}\,.$$

However

$$\begin{aligned}\|(D_t + P_j)v_j\|_{L^\infty([0,1];L^2(\mathbb{R}^2))} &\approx 2^j\|(D_t + P_j)v_j\|_{L^\infty([0,1];H^{-1}(\mathbb{R}^2))} \\ &\lesssim 2^j\{\|(P_j - P)v_j\|_{L^\infty([0,1];H^{-1}(\mathbb{R}^2))} + \|(D_t + P)v_j\|_{L^\infty([0,1];H^{-1}(\mathbb{R}^2))}\} \\ &\lesssim 2^j\|u_k\|_{L^\infty([0,1]\ L^2(M))}.\end{aligned} \tag{2.8}$$

The first line is due to the localization of P_j and v_j. Next we note that multiplication by a Lipschitz function ρ is a bounded operator in H^{-1}. Thus we regard P and P_j as in divergent form, we can thus bound the first term of the second line as (2.7). While the second term of the second line is also bounded, thanks again to Coifman-Meyer commutator theorem.

Combine (2.5) and (2.6), we thus have

$$\|(D_t+P)v_j\|_{L^\infty([0,1];L^2(\mathbb{R}^2))} \lesssim \min\{2^j\|u_k\|_{L^\infty([0,1]\ L^2(M)}, \|u_k\|_{L^\infty([0,1]\ H^1(M))}\}. \tag{2.9}$$

Now we are ready to handle the second term in in the right hand side of (2.4). For $j \le k$, we use

$$(2^j)^{s-1/3}\|(D_t+P)v_j\|_{L^\infty([0,1]\ L^2(\mathbb{R}^2))} \le (2^j)^{s-1/3}2^j\|u_k\|_{L^\infty([0,1]\ L^2(M))}.$$

Therefore the sum of $j=1,\cdots,k$ terms will be bounded by

$$C(2^k)^{1/3+\varepsilon}\|u_k\|_{L^\infty([0,1]\ L^2(M))}. \tag{2.10}$$

For $j \ge k$, we use

$$(2^j)^{s-1/3}\|(D_t+P)v_j\|_{L^\infty([0,1];L^2(\mathbb{R}^2))} \lesssim (2^{(j-k)})^{s-1/3}(2^{-k})^{1/3}\|\Lambda^s u_k\|_{L^\infty([0,1];H^1(M))}$$

Since $s-1/3<0$, the sum of $j \ge k$ terms is bounded by

$$(2^{-k})^{1/3}\|\Lambda^s u_k\|_{L^\infty([0,1]\ H^1(M))}. \tag{2.11}$$

Thus the sum of (2.10) and (2.11) is bounded by $\|\Lambda^s u_k\|_{L^\infty([0,1];H^1(M))}$.

Now let $\lambda = 2^j$, $w_\lambda = v_j$, (2.4) can be written as

$$\|w_\lambda\|_{L^2([0,1];L^\infty(\mathbb{R}^2))} \lesssim \lambda^{\frac{2}{3}+\varepsilon}(\|w_\lambda\|_{L^\infty([0,1];L^2(\mathbb{R}^2))} + \lambda^{-\frac{4}{3}}\|(D_t+P)w_\lambda\|_{L^\infty([0,1];L^2(\mathbb{R}^2))})$$

which is implied by showing for each interval I_λ with length $\lambda^{-\frac{4}{3}}$, we all have

$$\|w_\lambda\|_{L^2(I_\lambda;L^\infty(\mathbb{R}^2))} \lesssim (\lambda)^\varepsilon(\|w_\lambda\|_{L^\infty(I_\lambda;L^2(\mathbb{R}^2))} + \|(D_t+P)w_\lambda\|_{L^1(I_\lambda;L^2(\mathbb{R}^2))})$$

Recall that the operator P here is rough. Thus we regularize the coefficients of P by setting

$$g_\lambda^{ij} = S_{\lambda^{2/3}}(g^{ij}), \quad \rho_\lambda = S_{\lambda^{2/3}}(\rho)$$

where $S_{\lambda^{2/3}}$ denotes a truncation of a function to frequencies less than $\lambda^{\frac{2}{3}}$. Let P_λ be the operator with coefficients g_λ^{ij} and ρ_λ. Then

$$\|(P-P_\lambda)w_\lambda\|_{L^1(I_\lambda;L^2(\mathbb{R}^2))} \lesssim \|w_\lambda\|_{L^\infty(I_\lambda;L^2(\mathbb{R}^2))}$$

since we know

$$|g_\lambda^{ij} - g^{ij}| \lesssim \lambda^{-\frac{2}{3}}$$

and similarly for ρ.

Then we rescale the problem by letting $\mu = \lambda^{\frac{2}{3}}$ and define

$$u_\mu(t,x) = w_\lambda(\lambda^{-\frac{2}{3}}t, \lambda^{-\frac{1}{3}}x), \quad Q_\mu = P_\lambda(\lambda^{-\frac{1}{3}}x, D)$$

The function $u_\mu(t,\dot{)}$ is localized to frequencies of size μ, and the coefficients of Q_μ are localized to frequencies of the size less than $\mu^{\frac{1}{2}}$. This implies the following estimates of the coefficients of Q_μ

$$\|\partial_x^\alpha g_\lambda^{ij}(\lambda^{-\frac{1}{3}}x)\| + \|\partial_x^\alpha \rho_\lambda(\lambda^{-\frac{1}{3}}x)\| \le C_\alpha \mu^{\frac{1}{2}\max(0,|\alpha|-2)}.$$

The time interval I_λ scales to μ^{-1}. Also note that by our reduction $\|g_\lambda^{ij} - \delta^{ij}\|_{C^2} \ll 1$. Thus we have reduced the proof of Theorem 1.1 to the following

Theorem 2.1. *Suppose that $u(t,x)$ is localized to frequencies $|\xi| \in [\frac{1}{4}\lambda, 4\lambda]$ and solves*

$$(D_t + \sum_{1\leq i,j\leq n} a^{ij}(x)\partial_{x_i}\partial_{x_j} + \sum_{1\leq i\leq n} b^i(x)\partial_{x_i})u = F$$

Assume also that the metric satisfies

$$\|a^{ij} - \delta_{ij}\|_{C^2} \ll 1, \qquad \|b^i\|_{C^1} \lesssim 1$$

$$\operatorname{supp}(\widehat{a^{ij}}),\ \operatorname{supp}(\widehat{b^i}) \subset B_{\lambda^{1/2}}(0).$$

Then the following estimate holds

$$\|u\|_{L^2([0,\lambda^{-1}];L^\infty(\mathbb{R}^2))} \lesssim (\log\lambda)^{\frac{1}{2}}(\|u\|_{L^\infty([0,\lambda^{-1}];L^2(\mathbb{R}^2))} + \|F\|_{L^1([0,\lambda^{-1}];L^2(\mathbb{R}^2))})$$

3. Wave Packet and Parametrix

To prove Theorem 2.1, we need some notations for wave packet transform. We fix a real, radial Schwartz function $g(x) \in \mathcal{S}(\mathbb{R}^2)$, with $\|g\|_{L^2} = (2\pi)^{-1}$, and assume its Fourier transform $h(\xi) = \hat{g}(\xi)$ is supported in the unit ball $\{|\xi| < 1\}$. For $\lambda \geq 1$, we define $T_\lambda : \mathcal{S}'(\mathbb{R}^2) \to \mathcal{C}^\infty(\mathbb{R}^4)$by

$$(T_\lambda f)(x,\xi) = \lambda^{\frac{1}{2}} \int e^{-i\langle \xi, z-x\rangle} g(\lambda^{\frac{1}{2}}(z-x))f(z)dz.$$

A simple calculation shows that

$$f(y) = \lambda^{\frac{1}{2}} \int e^{i\langle \xi, y-x\rangle} g(\lambda^{\frac{1}{2}}(y-x))(T_\lambda f)(x,\xi)dxd\xi,$$

so that $T_\lambda^* T_\lambda = I$. In particular,

$$\|T_\lambda f\|_{L^2(\mathbb{R}^4_{x,\xi})} = \|f\|_{L^2(\mathbb{R}^2_x)}.$$

Let

$$D_t + A(x,D) + B(x,D) = D_t + \sum_{1\leq i,j\leq n} a^{ij}(x)\partial_{x_i}\partial_{x_j} + \sum_{1\leq i\leq n} b^i\partial_{x_i}.$$

We conjugate $A(x,D)$ by T_λ and take a suitable approximation to the resulting operator. Define the following differential operator over (x,ξ)

$$\widetilde{A} = -id_\xi a(x,\xi)\cdot d_x + id_x a(x,\xi)\cdot d_\xi + a(x,\xi) - \xi\cdot d_\xi a(x,\xi)$$

By the argument from wave packet methods (Lemmas 3.1-3.3 in Smith [19]), we have that if $\widetilde{\beta}_\lambda$ is a Littlewood-Paley cutoff truncating to frequencies $|\xi| \approx \lambda$ then

$$\|T_\lambda A(\cdot,D)\widetilde{\beta}_\lambda(D) - \widetilde{A}T_\lambda\widetilde{\beta}_\lambda(D)\|_{L^2_x\to L^2_{x,\xi}} \lesssim \lambda$$

This yields that, if $\tilde{u}(t,x,\xi) = (T_\lambda u(t,\cdot))(x,\xi)$, then $\tilde{u}$ solves the equation

$$(\partial_t + d_\xi a(x,\xi)\cdot d_x - d_x a(x,\xi)\cdot d_\xi + ia(x,\xi) - i\xi\cdot d_\xi a(x,\xi))\tilde{u}(t,x,\xi) = \tilde{G}(t,x,\xi)$$

where $\tilde{G}$ satisfies

$$\int_0^{\lambda^{-1}} \|\tilde{G}(t,x,\xi)\|_{L^2_{x,\xi}}dt \lesssim \|u\|_{L^\infty([0,\lambda^{-1}];L^2])} + \|F\|_{L^1([0,\lambda^{-1}];L^2)}$$

Given an integral curve $\gamma(r) \in \mathbb{R}^4_{x,\xi}$ of the vector field

$$\partial_t + d_\xi a(x,\xi) \cdot d_x - d_x a(x,\xi) \cdot d_\xi$$

with $\gamma(t) = (x,\xi)$, we denote $\chi_{s,t}(x,\xi) = (x_{s,t}, \xi_{s,t}) = \gamma(s)$. Also define

$$\sigma(x,\xi) = a(x,\xi) - \xi \cdot d_\xi a(x,\xi), \quad \psi(t,x,\xi) = \int_0^t \sigma(\chi_{r,t}(x,\xi))dr$$

This allows us to write

$$\tilde{u}(t,x,\xi) = e^{-i\psi(t,x,\xi)}\tilde{u}_0(\chi_{0,t}(x,\xi)) + \int_0^t e^{-i\psi(t-r,x,\xi)}\tilde{G}(r,\chi_{r,t}(x,\xi))dr$$

where $\tilde{u}$ is an integrable superposition over r of functions invariant under the flow of $\tilde{A}$, truncated to $t > r$.

Since $u(t,x) = T^*_\lambda \tilde{u}(t,x,\xi)$ it thus suffices to obtain estimates

$$\|\tilde{\beta}_\lambda(D)W_t f\|_{L^2_t L^\infty_x} \lesssim (\log \lambda)^{\frac{1}{2}} \|f\|_{L^2_{x,\xi}} \tag{3.1}$$

where W_t acts on function $f(x,\xi)$ by the formula

$$(W_t f)(y) = T^*_\lambda(e^{-i\psi(t,x,\xi)} f(\chi_{0,t}(\cdot)))(y) \tag{3.2}$$

In order to get the desired estimates by TT^* method, we investigate the kernel $K(t,y,s,x)$ of $W_t W_s^*$ which is

$$\lambda \int e^{-i\langle\zeta, x-z\rangle - i\int_s^t \sigma(\chi_{r,t}(z,\zeta)) + i\langle\zeta_t, y-z_t\rangle} g(\lambda^{\frac{1}{2}}(y - z_{t,s}))g(\lambda^{\frac{1}{2}}(x-z))dzd\zeta$$

Recall that $\operatorname{supp}(\hat{g}) \subset B_1(0)$. We are concerned with $\tilde{\beta}_\lambda W_t W_s^* \tilde{\beta}_\lambda$, thus we can inserted a cutoff $S_\lambda(\zeta)$ into the integrand which is supported in a set $|\zeta| \approx \lambda$. Also note that the Hamiltonian vector field is independent of time, that is $\chi_{t,s} = \chi_{t-s,0}$. We denote it by $\chi_{t-s,0}(z,\zeta) = \chi_{t-s}(z,\zeta) = (z_{t-s}, \zeta_{t-s})$. It then suffices to consider $s = 0$, and the kernel $K(t,x,0,y)$ as

$$\lambda \int e^{-i\langle\zeta, x-z\rangle - i\psi(t,z,\zeta) + i\langle\zeta_t, y-z_t\rangle} g(\lambda^{\frac{1}{2}}(y - z_{t,s}))g(\lambda^{\frac{1}{2}}(x-z))S_\lambda(\zeta)dzd\zeta$$

We will built the estimates (3.1) by considering the estimate for time variable between $[0, \lambda^{-2}]$ and $[\lambda^{-2}, \lambda^{-1}]$ respectively. That is we will prove

$$\|\tilde{\beta}_\lambda(D)W_t f\|_{L^2([0\ ,\ \lambda^{-2}];L^\infty(\mathbb{R}^2))} \lesssim \|f\|_{L^2_{x,\xi}} \tag{3.3}$$

and

$$\|\tilde{\beta}_\lambda(D)W_t f\|_{L^2([\lambda^{-2}\ ,\ \lambda^{-1}];L^\infty(\mathbb{R}^2))} \lesssim (\log \lambda)^{\frac{1}{2}} \|f\|_{L^2_{x,\xi}} \tag{3.4}$$

The inequality (3.3) is easy to prove , note that when $t \in [0, \lambda^{-2}]$, it is easy to see that

$$|K(t,x,0,y)| \approx \lambda \cdot (\lambda^{-\frac{1}{2}})^2 \cdot \lambda^2 = \lambda^2. \tag{3.5}$$

The term $(\lambda^{-\frac{1}{2}})^2$ came from the size of g and λ^2 from S_λ. Then the estimates follows from applying Schwartz inequality to time variables.

The inequality (3.4) comes from establishing

$$|K(t,x,0,y)| \lesssim \frac{1}{t} \tag{3.6}$$

for $t \in [\lambda^{-2}, \varepsilon\lambda^{-1}]$ with ε chosen sufficient small and independent of λ. Then by Schwartz inequality, we get

$$\|\tilde{\beta}_\lambda W_t W_s^* \tilde{\beta}_\lambda\|_{L^2 \to L^2} \lesssim \int_{\lambda^{-2}}^{\lambda^{-1}} \frac{1}{t} dt = \log \lambda.$$

The dispersive estimate (3.6) we need is actually proved in the section 4 of Blair, Smith and Sogge [5]. Hence we conclude Theorem 2.1.

4. Gradient Estimates

Next we will prove Theorem 1.3. Recall that we assume

$$\mathbb{I}_{\Lambda \leq \sqrt{-\Delta} \leq 2\Lambda}(f) = f \ \ , \ \ \mathbb{I}_{\Gamma \leq \sqrt{-\Delta} \leq 2\Gamma}(g) = g.$$

If $\Lambda > \Gamma$, we can prove as following

$$\begin{aligned}
\|\nabla(e^{it\triangle} f) e^{it\triangle} g\|_{L^2([0,1]\times M)} &\lesssim \|\nabla e^{it\triangle} f\|_{L^\infty([0,1];L^2(M))} \|e^{it\triangle} g\|_{L^2([0,1] L^\infty(M))} \\
&\lesssim \Lambda \|e^{it\triangle} f\|_{L^\infty([0,1];L^2(M))} \Gamma^s \|g\|_{L^2(M)} \\
&\lesssim \Lambda \Gamma^s \|f\|_{L^2(M)} \|g\|_{L^2(M)},
\end{aligned}$$

where we have used the fact Riesz transform $\nabla(-\triangle)^{-1/2}$ is bounded on $L^2(M)$ (see [18]) and then apply Hörmander multiple theorem (see [27]) in the second inequality.

If $\Lambda < \Gamma$, as the reduction (2.2), Let $r = \frac{5}{3} + \varepsilon$, $s = r - 1$. Then we need to prove that

$$\|\nabla u\|_{L^2([0,1];L^\infty(M))} \lesssim \|\Lambda^s f\|_{H^1(M)}$$

is true. Again we write it as

$$\|\nabla u_k\|_{L^2([0,1];L^\infty(M))} \lesssim \|\Lambda^s u_k\|_{H^1(M)} \tag{4.1}$$

for denoting that it's frequency being localized to $\Lambda = 2^k$. By making use of the following inequality

$$\|\nabla u_k\|_{L^2([0,1];L^\infty(M))} \lesssim \Lambda \|u_k\|_{L^2([0,1];L^\infty(M))} \tag{4.2}$$

and estimate (2.3) we conclude the result.

To see (4.2) is true, we will use an argument concerning finite speed of propagation of wave equation (see for example [21], [27]) and the following gradient estimate of unit band spectral projection operator. The unit band spectral projection operator is defined as

$$\chi_\lambda f(x) = \sum_{\lambda \leq \lambda_k < \lambda+1} E_k f(x) = \sum_{\lambda \leq \lambda_k < \lambda+1} e_k(x) \int_M f(y) e_k(y) dy$$

Theorem 4.1 ([27] Theorem 1)**.** *Fix a compact Riemannian manifold (M, g) with boundary and* $\dim M = n$*, for both Dirichlet Laplacian and Neumann Laplacian on M, there is a uniform constant C such that*

$$\|\nabla \chi_\lambda f\|_{L^\infty(M)} \leq C\lambda^{(n+1)/2}\|f\|_{L^2(M)} \tag{4.3}$$

In fact, we are going to use it's dual form , that is

$$\|\chi_\lambda \nabla f\|_{L^2(M)} \leq C\lambda^{(n+1)/2}\|f\|_{L^1(M)} \tag{4.4}$$

Let $\{\beta_j\}_{j\geq 0}$ be a Littlewood-Paley partition on $\mathbb{R}$. Since Littlewood -Paley operator commutes with Schrodinger operator, estimate (4.2) will be a consequence of

$$\|\nabla \beta_k(D) f\|_{L^\infty(M)} \lesssim \lambda \|f\|_{L^\infty(M)} \tag{4.5}$$

where $2^k = \lambda$ and f is spectrally localized to on dyadic interval of order λ. However we should prove the following dual inequality

$$\|\beta_k(D)\nabla f\|_{L^1(M)} \lesssim \lambda \|f\|_{L^1(M)}, \tag{4.6}$$

since this implies (4.5).

Recall that $\beta_j(\cdot) = \beta(\frac{\cdot}{2^j})$, $j \geq 1$ for some $\beta \in C_0^\infty(1/2, 4)$. We may assume it is an even function on $\mathbb{R}$, otherwise we only need replace $\beta(t)$ by $\tilde{\beta(t)}$ where the even function $\tilde{\beta(t)} = \beta(t)$ for $t > 0$. Write

$$\beta(\frac{P}{\lambda})\nabla f(x) = \frac{1}{2\pi}\int_{\mathbb{R}} \lambda\widehat{\beta}(\lambda t)e^{itP}\nabla f(x)dt.$$

Note that proving (4.6) is equivalent to considering

$$T_\lambda(P)f(x) = \int_{\mathbb{R}} \lambda\widehat{\beta}(\lambda t)\cos tP\nabla f(x)dt,$$

and proving

$$\|T_\lambda(P)f\|_{L^1(M)} \lesssim \lambda\|f\|_{L^1(M)} \tag{4.7}$$

Here $P = \sqrt{-\triangle}$ and

$$\cos tP\nabla f(x) = \sum_{k=1}^{\infty} \cos t\lambda_k E_k(\nabla f)(x) = u(t,x)$$

is the cosine transform of ∇f. It is the solution of wave equation

$$(\partial_t^2 - \triangle_g)u = 0\ ,\ u(0,\cdot) = \nabla f\ ,\ u_t(0,\cdot) = 0.$$

In order to prove (4.7) , we shall use the finite propagation speed for solutions to the wave equation. Specifically, if ∇f is supported in a geodesic ball $B(x_0, R)$ centered at x_0 with radius R, then $x \longrightarrow \cos tP\nabla f$ vanishes outside of $B(x_0, 2R)$ if $0 \leq t \leq R$.

Let $1 = \eta(t) + \sum_{j=1}^\infty \rho(2^{-j}t)$ be a Littlewood-Paley partition of $\mathbb{R}$. Write $T_\lambda = T_\lambda^0 + T_\lambda^j$, here

$$T_\lambda^0(P)f = \int_{\mathbb{R}} \eta(\lambda t)\lambda\widehat{\beta}(\lambda t)\cos tP\nabla f dt \tag{4.8}$$

and

$$T_\lambda^j(P)f = \int_{\mathbb{R}} \rho(2^{-j}\lambda t)\lambda\widehat{\beta}(\lambda t)\cos tP\nabla f dt \tag{4.9}$$

We will prove $T_\lambda(P)$ satisfies (4.7) by showing $T_\lambda^0(P)$ and $\sum_{j\geq 1} T_\lambda^j(P)$ both satisfy (4.7).

Now

$$\begin{aligned}
T_\lambda^0(P)f(x) &= \int_{\mathbb{R}} \eta(\lambda t)\lambda\widehat{\beta}(\lambda t)\cos tP\nabla f(x)dt \\
&= \int_{\mathbb{R}} \eta(\lambda t)\lambda\widehat{\beta}(\lambda t) \sum_{\lambda\leq\lambda_k\leq 2\lambda} \cos t\lambda_k e_k(x) \int_M e_k(y)\nabla f(y)dydt \\
&= \int_M \{\int_{\mathbb{R}} \eta(\lambda t)\lambda\widehat{\beta}(\lambda t) \sum_{\lambda\leq\lambda_k\leq 2\lambda} \cos t\lambda_k e_k(x)e_k(y)dt\}\nabla f(y)dy \\
&= \int_M K_\lambda^0(x,y)f(y)dy
\end{aligned}$$

Because the finite propagation speed of the wave equation mentioned before implies that the kernel of the operator $K_\lambda^0(x,y)$ must satisfy

$$K_\lambda^0(x,y) = 0 \quad \text{if} \quad \operatorname{dist}(x,y) > 8\lambda^{-1},$$

since $\cos tP$ will have a kernel that vanishes on this set when t belongs to the support of the integral defining $K_\lambda^0(x,y)$. Because of this, in order to prove T_λ^0 satisfies (4.7), it suffices to show that for all geodesic balls $B_{\lambda,0}$ with radius $8\lambda^{-1}$ one has the bound

$$\left\|T_\lambda^0 f\right\|_{L^1(B_{\lambda,0})} \lesssim \lambda\|f\|_{L^1(M)}, \tag{4.10}$$

For the L^1 norm over $B_{\lambda,0}$. Also we want to use (4.3), so rewrite

$$\nabla f = \sum_l \nabla f_l = \sum_{l=\lambda}^{2\lambda-1} \chi_l \nabla f$$

with each ∇f_l being spectrally localized to unit band.

By using Cauchy-Schwartz inequality, (4.4), and orthogonality we find

$$\begin{aligned}
\left\|T_\lambda^0 f\right\|_{L^1(B_{\lambda,0})} &\leq 8\lambda^{-1}\left\|T_\lambda^0 f\right\|_{L^2(M)} \\
&\leq C\lambda^{-1}\{\sum_{l=\lambda}^{2\lambda}(\sup_{l\leq\lambda_k<l}|\beta(\frac{\lambda_k}{\lambda})|^2)\left\|\chi_l\nabla f\right\|_{L^2(M)}^2\}^{1/2} \\
&\leq C\lambda^{-1}\lambda^{1/2}\lambda^{3/2}\left\|f\right\|_{L^1(M)}.
\end{aligned} \tag{4.11}$$

Similar,

$$\begin{aligned}
T_\lambda^j f(x) &= \int_{\mathbb{R}} \rho(2^{-j}\lambda t)\lambda\widehat{\beta}(\lambda t)\cos tP\nabla f(x)dt \\
&= \int_M \{\int_{\mathbb{R}} \rho(2^{-j}\lambda t)\lambda\widehat{\beta}(\lambda t) \sum_{\lambda\leq\lambda_k\leq 2\lambda} \cos t\lambda_k e_k(x)e_k(y)dt\}\nabla f(y)dy \\
&= \int_M K_\lambda^j(x,y)f(y)dy
\end{aligned} \tag{4.12}$$

has the property that $K_\lambda^j(x,y) = 0$ if $\mathrm{dist(x,y)} \geq 8 \cdot 2^{j+1} \cdot \lambda^{-1}$. Note that the dyadic cutoff localizes to $|t| \approx \lambda^{-1}2^j$. Hence follows again (4.11) yields the bound $2^{j+1}\lambda^{-1}\lambda^{1/2}(\lambda t)^{-N}(\lambda^{3/2})\|f\|_1$ with N be a large enough positive integer. Here the term $2^{j+1}\lambda^{-1}$ comes from the volume of geodesic ball $B_{\lambda,j}$ with radius $8 \cdot 2^{j+1} \cdot \lambda^{-1}$, $(\lambda t)^{-N}$ from value of β. Thus we have

$$\|T_\lambda^j\|_{L^1(B_{\lambda,j})} \lesssim \lambda 2^{-jN}\|f\|_{L^1(M)}$$

which form a geometric series and thus the sum of $j = 1, \cdots, \infty$ terms enjoys the property (4.7).

5. Cubic NLS

5.1. **Cauchy Problem.** In the following, we establish the well-posedness of the cubic nonlinear Schrödinger equation in 2 dimensional compact manifolds (M, g) with boundary. The equations we are interested in is following.

$$\begin{cases} i\partial_t u + \triangle u &= \alpha|u|^2 u, \text{ on } \mathbb{R} \times M \\ u|_{t=0} &= u_0, \text{ on } M \\ u|_{\partial M} &= 0 \text{ (Dirichlet)}, \quad \text{(or)} \quad \mathrm{N_x \cdot \nabla u|_{\partial M}} = 0 \text{ (Neumann)} \end{cases} \tag{5.1}$$

where $\alpha = \pm 1$.

Definition 5.1. Let s be a real number. We shall say that the Cauchy problem (5.1) is uniformly well-posed in $H^s(M)$ if, for any bounded subset of $H^s(M)$, there exists $T > 0$ such that the flow map

$$u_0 \in C^\infty(M) \cap B \mapsto u \in C([-T, T], H^s(M))$$

is uniformly continuous when the source space is endowed with H^s norm, and when the target space is endowed with

$$\|u\|_{C_T H^s} = sup_{|t| \leq T}\|u(t)\|_{H^s(M)}$$

Let's state again our local well-posdness results Theorem 1.5.

Theorem 1.5. *If (M, g) is a 2 dimensional manifold with boundary, then the Cauchy problem for* (5.1) *is uniformly well-posed in $H^s(M)$ for every $s > \frac{2}{3}$.*

5.2. **Bourgain Spaces.** In order to prove the local well-posedness of cubic nonlinear Schrödinger equation on manifolds with boundary. We introduce Bourgain space $X^{s,b}$. Our definition follows from Burq, Gérard and Tzvetkov [12] using the spectral projectors on manifolds.

Let (e_k) be a $L^2(M)$ orthonormal basis of eigenfunctions of Dirichlet(or Neumann) Laplacian $-\triangle_g$ with eigenvalues μ_k^2, E_k be the orthogonal projector along e_k. The Sobolev space $H^s(M)$ is associated to $(I - \triangle)^{1/2}$, equipped with the norm

$$\|u\|^2_{H^s(M)} = \sum_k \langle\mu_k\rangle^{2s}\|E_k u\|^2_{L^2(M)}$$

where $\langle\mu_k\rangle = (1 + \mu_k^2)^{\frac{1}{2}}$.

Definition 5.2. The space $X^{s,b}(\mathbb{R}\times M)$ is the completion of $C_0^\infty(\mathbb{R}_t;H^s(M))$ with the norm

$$\|u\|^2_{X^{s,b}(\mathbb{R}\times M)} = \sum_k \|\langle\tau+\mu_k^2\rangle^b\langle\mu_k\rangle^s\widehat{E_k u}(\tau)\|^2_{L^2(\mathbb{R}_\tau;L^2(M))} \tag{5.2}$$

$$= \|e^{-it\triangle}u(t,\cdot)\|^2_{H^b(R_t;H^s(M))} \tag{5.3}$$

where $\widehat{E_k u}(\tau)$ denote the Fourier transform of $E_k u$ with respect to the time variable.

In fact, if $s\geq 0$ and $u\in\mathcal{S}'(\mathbb{R},L^2(M))$. Let $F(t,\cdot)=e^{-it\triangle}u(t,\cdot)$, then $F(t,\cdot)\in\mathcal{S}'(\mathbb{R},L^2(M))$ and $E_k(F(t,\cdot))=e^{it\mu_k^2}E_k(u(t,\cdot))$. Hence $\widehat{E_k(F)}(\tau)=\widehat{E_k(u)}(\tau-\mu_k^2)$. Applies this to (5.2) , we conclude

$$\|u\|^2_{X^{s,b}(\mathbb{R}\times M)} = \|e^{-it\triangle}u(t,\cdot)\|^2_{H^b(R_t;H^s(M))}.$$

We also note that if $b>\frac{1}{2}$, $H^b(\mathbb{R},H^s(M))\hookrightarrow C(\mathbb{R},H^s(M))$, since $u(t,\cdot)=e^{it\triangle}F(t,\cdot)$, we have $u\in C(\mathbb{R},H^s(M))$.

In order to use a contraction mapping argument to obtain local existence. We need to define local in time version of $X^{s,b}(\mathbb{R}\times M)$. For $T>0$ we denoted by $X_T^{s,b}(M)$ the space of restrictions of elements of $X^{s,b}(\mathbb{R}\times M)$ endowed with the norm

$$\|u\|_{X_T^{s,b}} = \inf\{\|\tilde{u}\|_{X^{s,b}(\mathbb{R}\times M)}\ ,\ \tilde{u}|_{(-T,T)\times M}=u\}$$

Now we can reformulate the bilinear estimates in the $X^{s,b}$ content. The following lemma should refer to the lemma 2.3 of [12].

Lemma 5.3. *Let $s\in\mathbb{R}$. The following statements are equivalent:*

(1) For any u_0 , $v_0\in L^2(M)$ satisfying

$$1_{\lambda\leq\sqrt{-\triangle}\leq 2\lambda}u_0=u_0 \quad , \quad 1_{\mu\leq\sqrt{-\triangle}\leq 2\mu}v_0=v_0$$

one has

$$\|e^{it\triangle}u_0\, e^{it\triangle}v_0\|_{L^2((0,1)_t\times M)} \leq C(\min(\lambda,\mu))^s\|u_0\|_{L^2(M)}\|v_0\|_{L^2(M)} \tag{5.4}$$

(2)For any $b>\frac{1}{2}$ and any $f,g\in X^{0,b}(\mathbb{R}\times M)$ satisfying

$$1_{\lambda\leq\sqrt{-\triangle}\leq 2\lambda}f=f \quad , \quad 1_{\mu\leq\sqrt{-\triangle}\leq 2\mu}g=g$$

one has

$$\|fg\|_{L^2(\mathbb{R}\times M)} \leq C(\min(\lambda,\mu))^s\|f\|_{X^{0,b}(\mathbb{R}\times M)}\|g\|_{X^{0,b}(\mathbb{R}\times M)} \tag{5.5}$$

Proof. If $u(t)=e^{-it\triangle}u_0$ then for any $\psi\in C_0^\infty(\mathbb{R})$ and any b , $\psi(t)u(t)\in X^{0,b}(\mathbb{R}_t\times M)$ with

$$\|\psi u\|_{X^{0,b}(\mathbb{R}\times M)} \leq C\|u_0\|_{L^2(M)}$$

which shows that (5.5) implies (5.4).

Suppose that $f(t)$ and $g(t)$ are supported in time in the interval $(0,1)$ and write

$$f(t)=e^{it\triangle}e^{-it\triangle}f(t)=e^{it\triangle}F(t)\ ,\ g(t)=e^{it\triangle}e^{-it\triangle}g(t)=e^{it\triangle}G(t)$$

Then

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{it\tau}e^{it\triangle}\widehat{F}(\tau)d\tau \ , \ g(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{it\tau}e^{it\triangle}\widehat{G}(\tau)d\tau$$

and hence

$$(fg)(t) = \frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{it(\tau+\sigma)}e^{it\triangle}\widehat{F}(\tau)e^{it\triangle}\widehat{G}(\sigma)d\tau d\sigma.$$

Ignoring the oscillating factors $e^{it(\tau+\sigma)}$, using (5.4) and the Cauchy-Schwartz inequality in (τ,σ) (in this places we use that $b > \frac{1}{2}$ to get the needed integrability) yields

$$\begin{aligned}\|fg\|_{L^2((0,1)\times M)} &\le C(\min(\lambda,\mu))^s \int_{\tau,\sigma} \|\widehat{F}(\tau)\|_{L^2(M)}\|\widehat{G}(\sigma)\|_{L^2(M)}d\tau d\sigma \\ &\le C(\min(\lambda,\mu))^s\|\langle\tau\rangle^b\widehat{F}(\tau)\|_{L^2(\mathbb{R}_\tau\times M)}\|\langle\sigma\rangle^b\widehat{G}(\sigma)\|_{L^2(\mathbb{R}_\sigma\times M)} \\ &= C(\min(\lambda,\mu))^s\|f\|_{X^{0,b}(\mathbb{R}\times M)}\|g\|_{X^{0,b}(\mathbb{R}\times M)}\end{aligned} \tag{5.6}$$

Finally, by decomposing $f(t) = \sum_{n\in\mathbb{Z}}\psi(t-\frac{n}{2})f(t)$ and $g(t) = \sum_{n\in\mathbb{Z}}\psi(t-\frac{n}{2})g(t)$ with a suitable $\psi \in C_0^\infty(\mathbb{R})$ supported in (0,1), the general case for $f(t)$ and $g(t)$ follows from the considered particular case of $f(t)$ and $g(t)$ supported in time in the interval $(0,1)$. Thus (5.4) implies (5.5). □

A similar proof for the gradient bilinear estimates should refer to Anton [3].

Lemma 5.4. *Let $s \in \mathbb{R}$. The following statements are equivalent:*

(1) For any u_0 , $v_0 \in L^2(M)$ satisfying

$$1_{\lambda\le\sqrt{-\triangle}\le 2\lambda}u_0 = u_0 \quad , \quad 1_{\mu\le\sqrt{-\triangle}\le 2\mu}v_0 = v_0$$

one has

$$\|(\nabla e^{it\triangle}u_0)\ e^{it\triangle}v_0\|_{L^2((0,1)_t\times M)} \le C\lambda(\min(\lambda,\mu))^s\|u_0\|_{L^2(M)}\|v_0\|_{L^2(M)} \tag{5.7}$$

(2)For any $b > \frac{1}{2}$ and any $f, g \in X^{0,b}(\mathbb{R}\times M)$ satisfying

$$1_{\lambda\le\sqrt{-\triangle}\le 2\lambda}f = f \quad , \quad 1_{\mu\le\sqrt{-\triangle}\le 2\mu}g = g$$

one has

$$\|(\nabla f)g\|_{L^2(\mathbb{R}\times M)} \le C\lambda(\min(\lambda,\mu))^s\|f\|_{X^{0,b}(\mathbb{R}\times M)}\|g\|_{X^{0,b}(\mathbb{R}\times M)} \tag{5.8}$$

Denote by $S(t) = e^{it\triangle}$ the free evolution. Using the Duhamel formula , we know that to solve (5.1) is equivalent to solve the integral equation

$$u(t) = S(t)u_0 - i\alpha\int_0^t S(t-\tau)\{|u(\tau)|^2u(\tau)\}d\tau$$

To deal with it , we need the following lemmas:

Lemma 5.5. *Let b , $s > 0$ and let $u_0 \in H^s(M)$. Then*

$$\|S(t)u_0\|_{X_T^{s,b}} \lesssim T^{\frac{1}{2}-b}\|u_0\|_{H^s} \tag{5.9}$$

Lemma 5.6. *Let $0 < b' < \frac{1}{2}$ and $0 < b < 1 - b'$. Then for all $F \in X_T^{s,-b'}(M)$,*

$$\| \int_0^t S(t-\tau)F(\tau)d\tau \|_{X_T^{s,b}(M)} \lesssim T^{1-b-b'} \|F\|_{X_T^{s,-b'}(M)} \tag{5.10}$$

Lemma 5.7. *For $s > s_0$, there exists $(b, b') \in \mathbb{R}^2$, satisfying*

$$0 < b' < \frac{1}{2} < b \ , \ \ b + b' < 1, \tag{5.11}$$

and $C > 0$ such that for every triple $(u_j), j = 1, 2, 3$ in $X^{s,b}(\mathbb{R} \times M)$

$$\|u_1 u_2 u_3\|_{X^{s,-b'}(\mathbb{R}\times M)} \leq C \prod_{j=1}^{3} \|u_j\|_{X^{s,b}(\mathbb{R}\times M)}. \tag{5.12}$$

Lemma 5.5 is easy to see.

Proof. Let $\varepsilon > 0$ and $\varphi \in C_0^\infty(\mathbb{R})$, $\varphi = 1$ on $(-T-\varepsilon, T+\varepsilon)$. Then $\|S(t)u_0\|_{X_T^{s,b}} \leq \|\varphi(t)S(t)u_0\|_{X^{s,b}} \leq \|\varphi(t)u_0\|_{H^b(\mathbb{R},H^s(M))} \leq cT^{\frac{1}{2}-b}\|u_0\|_{H^s(M)}$. □

The lemma 5.6 is due to Bourgain [7], we also refer to Ginibre [15] for a simpler proof.

The proof of lemma 5.7 will rely on the bilinear estimates (5.5) and (5.8). However we will postpone this proof and see how can we proof theorem 1.5 by these there lemmas first.

Proof. (of Theorem 1.5) To solve NLS equation is equivalent to solve the integral equation with Dirichlet (or Neumann) boundary conditions

$$u(t) = S(t)u_0 - i\alpha \int_0^t S(t-\tau)\{|u(\tau)|^2 u(\tau)\}d\tau$$

We denote by $\Phi(u)$ by the left hand side of the equation.

Consider $(b, b') \in \mathbb{R}^2$ given by lemma 5.6 and let $R > 0$ and $u_0 \in H^s(M)$ such that $\|u_0\|_{H^s} \leq R$. We show that there exists $R' > 0$ and $0 < T < 1$ depending on R such that Φ is a contracting map from the ball $B(0, R') \subset X_T^{s,b}(M)$ onto itself.

From the linear estimate (5.9) we know that $\|S(t)u_0\|_{X_1^{s,b}(M)} \leq c\|u_0\|_{H^s}$. From the definition of $X_T^{s,b}$ spaces we know that $T_1 < T_2$ implies $X_{T_2}^{s,b} \subset X_{T_1}^{s,b}$. Therefore for $T < 1$, $\|S(t)u_0\|_{X_T^{s,b}(M)} \leq c_0\|u_0\|_{H^s}$.

Define $R' = 2c_0 R$. From estimates (5.10) , we obtain for $T < 1$,

$$\|\Phi(u)\|_{X_T^{s,b}(M)} \leq c_0\|u_0\|_{H^s} + c_1 T^{1-b-b'} \|u\overline{u}u\|_{X_T^{s,-b'}(M)}$$

Combine this with (5.12) gives

$$\|\Phi(u)\|_{X_T^{s,b}(M)} \leq c_0\|u_0\|_{H^s} + c_2 T^{1-b-b'} \|u\|^3_{X_T^{s,b}(M)}.$$

Taking $T < 1$ such that $T^{1-b-b'} c_2 R'^3 \le c_0 R$, we ensure $\Phi : B(0,R') \subset X_T^{s,b} \to B(0,R') \subset X_T^{s,b}$. In addition Φ is a contraction, let $u_1, u_2 \in B(0,R') \subset X_T^{s,b}$, then

$$\|\Phi(u_1) - \Phi(u_2)\|_{X_T^{s,b}(M)} \le c_2 T^{1-b-b'} \||u_1|^2 u_2 - |u_2|^2 u_1\|_{X_T^{s,b}(M)}.$$

Using the decomposition $|u_1|^2 u_1 - |u_2|^2 u_2 = u_1^2(\overline{u}_1 - \overline{u}_2) + \overline{u}_2(u_1 - u_2)(u_1 + u_2)$, (5.10) and (5.12) , we get

$$\|\Phi(u_1) - \Phi(u_2)\|_{X_T^{s,b}(M)} \le c_3 T^{1-b-b'} R'^2 \|u_1 - u_2\|_{X_T^{s,b}(M)}.$$

By choosing $T < 1$ sufficient small , we know Φ is a contraction. Thus there exists an uniqueness $u \in X_T^{s,b}(M)$ such that $\Phi(u) = u$. Since $b > \frac{1}{2}$, $u \in C((-T,T), H^s(M))$. The flow $u_0 \in B(0,R) \subset H^s(M) \to u \in X_T^{s,b}(M)$ is Lipschitz. For if u , v are two solutions with initial data u_0 , v_0, we have as above

$$\|u - v\|_{X_T^{s,b}} \le c\|u_0 - v_0\|_{H^s} + c_3 T^{1-b-b'} R'^2 \|u - v\|_{X_T^{s,b}}.$$

By choosing T small enough , we have

$$\|u - v\|_{X_T^{s,b}} \le c\|u_0 - v_0\|_{H^s}$$

□

5.3. **Nonlinear Analysis.** Now we only owe to prove Lemma 5.7. We will use a decomposition of the spectrum of functions $u_j \in X^{s,b}(\mathbb{R} \times M)$.

The duality argument leads to the following equivalence: $u \in X^{s,b}(\mathbb{R} \times M)$, $\Leftrightarrow$ for all $u_0 \in X^{\infty,\infty}(\mathbb{R} \times M) = \cap_{s>0, b\in\mathbb{R}} X^{s,b}(\mathbb{R} \times M)$ we have

$$| < u, u_0 > | \le c\|u_0\|_{X^{-s,-b}(\mathbb{R}\times M)}$$

where $<,>$ denote the bracket pairing $\mathcal{S}'$ and $\mathcal{S}$. Thus (5.12) is implied by

$$| \int_{\mathbb{R}} \int_M u_0 u_1 u_2 u_3 dx dt| \le c \prod_{j=1}^{3} \|u_j\|_{X^{s,b}(\mathbb{R}\times M)} \|u_0\|_{X^{-s,b'}(\mathbb{R}\times M)} \tag{5.13}$$

holding for all $u_0 \in X^{\infty,\infty}(\mathbb{R} \times M)$. We will prove a similar result for spectrally localized functions and then sum over all frequencies.

For $j \in \{0,1,2,3\}$ and $N_j \in 2^{\mathbb{N}}$. We denote by $u_{jN_j} = 1_{\sqrt{-\triangle}\in[N_j,2N_j]} u_j$. Using the definition of $X^{s,b}(\mathbb{R} \times M)$ spaces the following equivalence holds

$$\|u_j\|^2_{X^{s,b}(\mathbb{R}\times M)} \cong \sum_{N_j \in 2^{\mathbb{N}}} \|u_{jN_j}\|^2_{X^{s,b}(\mathbb{R}\times M)} \cong \sum_{N_j \in 2^{\mathbb{N}}} N_j^{2s} \|u_{jN_j}\|^2_{X^{0,b}(\mathbb{R}\times M)}. \tag{5.14}$$

We denote by $\underline{N} = (N_0, N_1, N_2, N_3)$ the quadruple of 2^n numbers, $n \in \mathbb{N}$. Also

$$I(\underline{N}) = \int_{\mathbb{R}\times M} \prod_{i=0}^{3} u_{jN_j} dx dt$$

In order to prove Lemma 5.7. We need the two estimates about $I(\underline{N})$ in the following lemma. The proof of first estimate is standard by using (5.5) , while the second estimate in this lemma with Dirichlet boundary condition was proved by Anton [2] using (5.8). The same argument works for either Dirichlet or Neumann

condition. For the completeness and benefit of readers to understand how the bilinear estimates and gradient bilinear estimates working in nonlinear analysis, we include its proof here .

We also need the fact that

$$\|f\|_{L^4(\mathbb{R},L^2(M))} \leq \|f\|_{X^{0,\frac{1}{4}}(\mathbb{R}\times M)}. \tag{5.15}$$

This is due to conservation of L^2 norm by the linear Schrödinger flow and Sobolev embedding $H^{\frac{1}{4}}(\mathbb{R}) \hookrightarrow L^4(\mathbb{R})$, thus

$$\|f\|_{L^4(\mathbb{R},L^2(M))} = \|e^{it\triangle}f\|_{L^4(\mathbb{R},L^2(M))} \leq \|e^{it\triangle}f\|_{H^{\frac{1}{4}}(\mathbb{R}\times L^2(M))} = \|f\|_{X^{0,\frac{1}{4}}(\mathbb{R}\times M)}.$$

Lemma 5.8. *If* (5.4) *and* (5.7) *hold for $s > s_0$, then for all $s' > s_0$ there exists $0 < b' < \frac{1}{2}$, $c > 0$ such that, assuming $N_3 \leq N_2 \leq N_1$, the following estimates hold:*

$$|I(\underline{N})| \leq c(N_2N_3)^{s'} \prod_{j=0}^{3} \|u_{jN_j}\|_{X^{0,b'}(\mathbb{R}\times M)} \tag{5.16}$$

$$|I(\underline{N})| \leq c(\frac{N_1}{N_0})^2(N_2N_3)^{s'} \prod_{j=0}^{3} \|u_{jN_j}\|_{X^{0,b'}(\mathbb{R}\times M)} \tag{5.17}$$

Proof. Use Holder inequality, we get

$$\begin{aligned} |I(\underline{N})| &\leq \|u_{3N_3}\|_{L^4(L^\infty_x)}\|u_{2N_2}\|_{L^4(L^\infty_x)}\|u_{1N_1}\|_{L^4(L^2_x)}\|u_{0N_0}\|_{L^4(L^2_x)} \\ &\leq c(N_2N_3)^{1+\varepsilon} \prod_{j=0}^{3} \|u_{jN_j}\|_{L^4(L^2_x)} \\ &\leq c(N_2N_3)^{1+\varepsilon} \prod_{j=0}^{3} \|u_{jN_j}\|_{X^{0,\frac{1}{4}}(\mathbb{R}\times M)} \end{aligned} \tag{5.18}$$

In the second inequality, we use Sobolev embedding $\|u_{N_j}\|_{L^\infty(M)} \leq cN_j^{1+\varepsilon}\|u_{N_j}\|_{L^2(M)}$. The third inequality came from (5.15) .

Use Cauchy inequality and (5.5) (which is implied by (5.4)), we obtain that for any $b_0 > \frac{1}{2}$ there exists $c_0 > 0$ such that

$$\begin{aligned} |I(\underline{N})| &\leq \|u_{0N_0}u_{2N_2}\|_{L^2(\mathbb{R}\times M)}\|u_{1N_1}u_{3N_3}\|_{L^2(\mathbb{R}\times M)} \\ &\leq c_1(N_2N_3)^{s_0} \prod_{j=0}^{3} \|u_{jN_j}\|_{X^{0,b_0}(\mathbb{R}\times M)} \end{aligned} \tag{5.19}$$

We need further decomposition $u_{jN_j} = \sum_{K_j} u_{jN_jK_j}$ for interpolation, where $u_{jN_jK_j} = 1_{K_j \leq \langle i\partial_t + \triangle\rangle \leq 2K_j} u_{jN_j}$ and the sum is taken over 2^n numbers , for $n \in \mathbb{N}$: $K_j \in 2^{\mathbb{N}}$. Let us denote $I(\underline{N},\underline{K}) = \int_{\mathbb{R}\times M} \prod_{j=0}^{3} u_{jN_jK_j}$. Estimates (5.18) and (5.19) give

$$|I(\underline{N},\underline{K})| \leq c(N_2N_3)^{\alpha}(\prod_{j=0}^{3} K_j)^{\beta} \prod_{j=0}^{3} \|u_{jN_jK_j}\|_{L^2(\mathbb{R}\times M)}$$

where (α,β) equals $(1+\varepsilon,\frac{1}{4})$ or (s_0,b_0). For $s_0<s<1$ we can choose $\varepsilon>0$, $b_0>\frac{1}{2}$ and $0<b_1<\frac{1}{2}$ such that by interpolation we have the same estimates for $(\alpha,\beta)=(s',b_1)$.

Taking $b'\in(b_1,\frac{1}{2})$, this reads

$$|I(\underline{N},\underline{K})|\le c(N_2N_3)^{s'}\prod_{j=0}^{3}K_j^{b_1-b'}\|u_{jN_jK_j}\|_{X^{0,b'}(\mathbb{R}\times M)}.$$

Summing up over $\underline{K}\in(2^{\mathbb{N}})^4$, by geometric series and using Cauchy Schwartz, we obtain

$$|I(\underline{N})|\le c(N_2N_3)^{s'}\prod_{j=0}^{3}\|u_{jN_j}\|_{X^{0,b'}(\mathbb{R}\times M)}$$

which conclude the proof of (5.16).

For the proof of (5.17) , we start with Green formula:

$$\int_M \triangle fg-f\triangle gdx=\int_{\partial M}\frac{\partial f}{\partial \upsilon}g-f\frac{\partial g}{\partial \upsilon}d\sigma$$

If e_k are eigenfunctions of the Dicichlet(or Neumann) Laplacian associated with eigenvalues λ_k^2. The $u_{0N_0}=\sum_{\lambda_k\sim N_0}c_ke_k$, where $c_k=(u_{0N_0},e_k)$. We write

$$u_{0N_0}=-\frac{\Delta}{N_0^2}\sum_{\lambda_k\sim N_0}c_k(\frac{N_0}{\lambda_k})^2e_k.$$

Define $Tu_{0N_0}=\sum_{\lambda_k\sim N_0}c_k(\frac{N_0}{\lambda_k})^2e_k$ and $Vu_{0N_0}=\sum_{\lambda_k\sim N_0}c_k(\frac{\lambda_k}{N_0})^2e_k$. Then we have $TVu_{0N_0}=VTu_{0N_0}=u_{0N_0}$ and $\|Tu_{0N_0}\|_{H^s}\sim\|u_{0N_0}\|_{H^s}$ for all s. Use this notation $u_{0N_0}=-\frac{\Delta}{(N_0)^2}Tu_{0N_0}$. Apply it to green formula and using $u_{jN_j}|_{\partial M}=0$ (or $N_x\cdot\nabla u|_{\partial M}=0$), we obtain

$$I(\underline{N})=\frac{1}{N_0^2}\int_{\mathbb{R}\times M}Tu_0N_0\Delta(u_{1N_1}u_{2N_2}u_{3N_3})$$

By Leibniz's law, we have to deal with summation of terms of the forms

$$\frac{1}{N_0^2}J_{11}(\underline{N})=\frac{1}{N_0^2}\int_{\mathbb{R}\times M}Tu_{0N_0}(\Delta u_{1N_1})u_{2N_2}u_{3N_3}$$

and

$$\frac{1}{N_0^2}J_{12}(\underline{N})=\frac{1}{N_0^2}\int_{\mathbb{R}\times M}Tu_{0N_0}(\nabla u_{1N_1})(\nabla u_{2N_2})u_{3N_3}.$$

As we will see soon, they are always the largest terms in each sum. Use $\triangle u_{2N_2}$ we get $J_{11}(\underline{N})=-N_1^2\int_{\mathbb{R}\times M}Tu_{0N_0}Vu_{1N_1}u_{2N_2}u_{3N_3}$. Thus by (5.16) and $\|u_{jN_j}\|_{H^s}\sim\|Tu_{jN_j}\|_{H^s}\sim\|Vu_{jN_j}\|_{H^s}$, we have

$$\frac{1}{N_0^2}|J_{11}(\underline{N})|\le c\frac{N_1^2}{N_0^2}(N_2N_3)^{s'}\prod_{j=0}^{3}\|u_{jN_j}\|_{X^{0,b'}}(\mathbb{R}\times M).$$

To estimates $J_{12}(\underline{N})$, we note that $\|\nabla u_{jN_j}\|_{L^2(M)} \leq cN_j\|u_{jN_j}\|_{L^2(M)}$. Use the same process as in the proof of (5.16) , then (5.18) and (5.19) correspond to

$$|J_{12}(\underline{N})| \leq c(N_1N_2)(N_2N_3)^{1+\varepsilon}\prod_{j=0}^{3}\|u_{jN_j}\|_{X^{0,\frac{1}{4}}(\mathbb{R})\times M}$$

and

$$|J_{12}(\underline{N})| \leq c(N_1N_2)(N_2N_3)^{s_0}\prod_{j=0}^{3}\|u_{jN_j}\|_{X^{0,b_0}(\mathbb{R})\times M}.$$

In fact, we just got an additional term N_1N_2 in these new estimates. Therefore the interpolation argument leads to

$$\frac{1}{N_0^2}|J_{12}(\underline{N})| \leq c\frac{N_1N_2}{N_0^2}(N_2N_3)^{s'}\prod_{j=0}^{3}\|u_{jN_j}\|_{X^{0,b'}}(\mathbb{R}\times M).$$

Since $N_1N_2 \leq N_1^2$, we are done. □

Now we can use Lemma 5.8 to prove Lemma 5.7 .

Proof. (Proof of Lemma (5.7))
Our goal is to prove (5.12) . Use the same notation as above, we consider $I(\underline{N}) = \int_{\mathbb{R}\times M}\prod_{i=0}^{3}u_{jN_j}dxdt$. Without loss of generality, we may assume $N_3 \leq N_2 \leq N_1$.

Let $\frac{2}{3} < s' < s$. Using (5.17) in Lemma 5.8 and (5.14) , we have

$$|\sum_{N_0<cN_1} I(\underline{N})| \leq c\sum_{N_0<cN_1}(N_2N_3)^{s'-s}(\frac{N_0}{N_1})^s\|u_{0N_0}\|_{X^{-s,b'}\mathbb{R}\times M}\prod_{j=1}^{3}\|u_{jN_j}\|_{X^{s,b'}(\mathbb{R}\times M)}.$$

Using Cauchy Schwartz inequality and (5.14), we have

$$|\sum_{N_0<cN_1} I(\underline{N})| \leq c\|u_2\|_{X^{s,b'}(\mathbb{R}\times M)}\|u_3\|_{X^{s,b'}(\mathbb{R}\times M)}\sum_{N_0\leq CN_1}(\frac{N_0}{N_1})^s\alpha(N_0)\beta(N_1).$$

where $\alpha(N_0) = \|u_{0N_0}\|_{X^{-s,b'}(\mathbb{R}\times M)}$ and $\beta(N_1) = \|u_{1N_1}\|_{X^{s,b'}(\mathbb{R}\times M)}$. Thus we have

$$\sum_{N_0}\alpha(N_0)^2 \cong \|u_0\|^2_{X^{-s,b'}} ,\ \sum_{N_1}\beta(N_1)^2 \cong \|u_1\|^2_{X^{s,b'}}.$$

Since N_0 , N_1 are both dyadic numbers, we write $N_1 = 2^lN_0$ and $N_0 \geq N(l) = \max(1, 2^{-l})$, where l is an integer, $l \geq -l_0$ for some $l_0 \in \mathbb{N}$ depending on c. Thus

$$\begin{aligned}\sum_{N_0<cN_1}(\frac{N_0}{N_1})^s\alpha(N_0)\beta(N_1) &= \sum_{l\geq -l_0}\sum_{N_0\geq N(l)}2^{-sl}\alpha(N_0)\beta(2^lN_0)\\ &\leq \sum_{l>-l_0}2^{-sl}(\sum_{N_0}\alpha(N_0)^2)^{\frac{1}{2}}(\sum_{N_0>N(l)}\beta(2^lN_0)^2)^{\frac{1}{2}}\\ &\leq c\|u_0\|_{X^{-s,b'}(\mathbb{R}\times M)}\|u_1\|_{X^{s,b'}(\mathbb{R}\times M)}\end{aligned}$$

Since $\|u\|_{X^{s,b'}} \leq \|u\|_{X^{s,b}}$ for $b' < b$, we conclude that

$$|\sum_{N_0 < cN_1} I(\underline{N})| \leq c\|u_0\|_{X^{-s,b'}} \prod_{j=1}^{3} \|u_j\|^{X^{s,b}}.$$

For $N_0 \geq cN_1$, we use (5.17) of Lemma 5.8 to get:

$$|\sum_{N_0 \geq cN_1} I(\underline{N})| \leq c \sum_{N_0 \geq cN_1} (N_2 N_3)^{s'-s} (\frac{N_1}{N_0})^{2-s} \|u_{0N_0}\|_{X^{-s,b'}\mathbb{R}\times M} \prod_{j=1}^{3} \|u_{jN_j}\|_{X^{s,b'}(\mathbb{R}\times M)}.$$

This is just an exchange the role of N_0 and N_1 in the previous argument. Thus we obtain again

$$|\sum_{N_0 \geq cN_1} I(\underline{N})| \leq c\|u_0\|_{X^{-s,b'}(\mathbb{R}\times M)} \|u_1\|_{X^{s,b'}(\mathbb{R}\times M)} \|u_2\|_{X^{s,b'}(\mathbb{R}\times M)} \|u_3\|_{X^{s,b'}(\mathbb{R}\times M)}$$

□

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INSTITUTE OF MATHEMATICS, ACADEMIA SINICA, TAIPEI, TAIWAN, 11529, R.O.C.

E-mail address: jiangjc@math.sinica.edu.tw